IOL & DISCOGA research seminar

Greedy Strategies for Exhaustive Generation

Arturo Merino

The Greedy Gray Code Algorithm

Aaron Williams

Department of Mathematics and Statistics, McGill University
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Abstract. We reinterpret classic Gray codes for binary strings, permutations, combinations, binary trees, and set partitions using a simple greedy algorithm. The algorithm begins with an initial object and an ordered list of operations, and then repeatedly creates a new object by applying the first possible operation to the most recently created object.

1 Introduction

Let $\mathbb{B}(n)$ be the set of n-bit binary strings. The *binary reflected Gray code* **Gray**(n) orders $\mathbb{B}(n)$ so that successive strings have Hamming distance one (i.e., they differ in one bit). For example, the order for $n = 3$ appears below.

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COMBINATORIAL GENERATION VIA PERMUTATION LANGUAGES. I. FUNDAMENTALS

ELIZABETH HARTUNG, HUNG P. HOANG, TORSTEN MÜTZE, AND AARON WILLIAMS

ABSTRACT. In this work we present a general and versatile algorithmic framework for exhaustively generating a large variety of different combinatorial objects, based on encoding them as permutations. This approach provides a unified view on many known results and allows us to prove many new ones. In particular, we obtain the following four classical Gray codes as special cases: the Steinhaus-Johnson-Trotter algorithm to generate all permutations of an n-element set by adjacent transpositions; the binary reflected Gray code to generate all n-bit strings by flipping a single bit in each step; the Gray code for generating all n-vertex binary trees by rotations due to Lucas, van Baronaigien, and Ruskey; the Gray code for generating all partitions of an n-element ground set by element exchanges due to Kaye.

We present two distinct applications for our new framework: The first main application is the generation of pattern-avoiding permutations, yielding new Gray codes for different families of permutations that are characterized by the avoidance of certain classical patterns, (bi)vincular patterns, barred patterns, boxed patterns, Bruhat-restricted patterns, mesh patterns, monotone and geometric grid classes, and many others. We also obtain new Gray codes for all the combinatorial objects that are in bijection to these permutations, in particular for five different types of geometric rectangulations, also known as floorplans, which are divisions of a square into n rectangles subject to certain restrictions.

The second main application of our framework are lattice congruences of the weak order on the symmetric group S_n. Recently, Pilaud and Santos realized all those lattice congruences as $(n-1)$-dimensional polytopes, called quotientopes, which generalize hypercubes, associahedra, permutahedra etc. Our algorithm generates the equivalence classes of each of those lattice congruences, by producing a Hamilton path on the skeleton of the corresponding quotientope,

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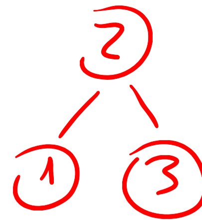

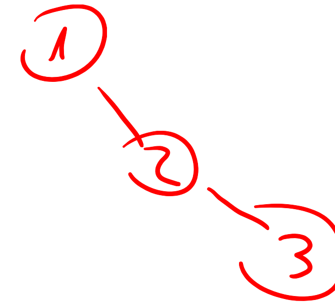

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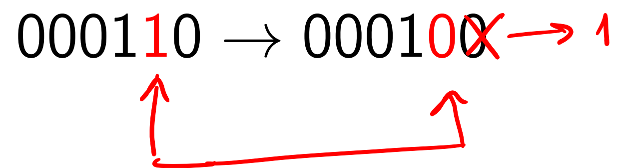

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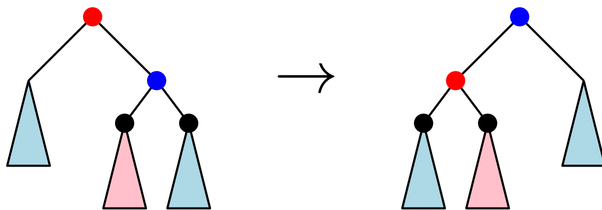

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    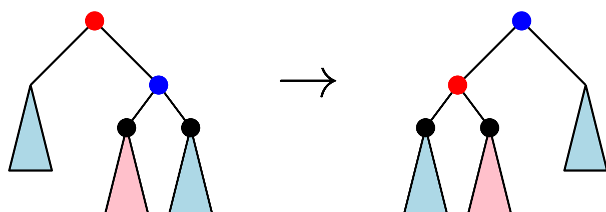

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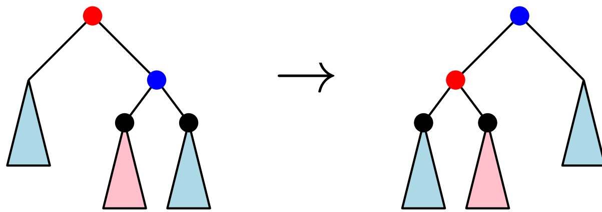

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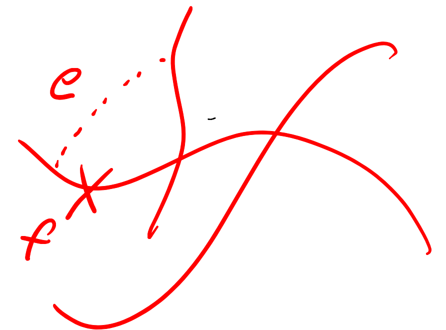

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    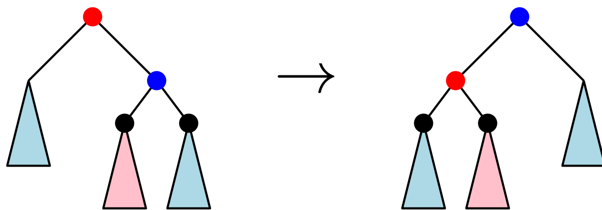

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- A first step towards efficient generation.

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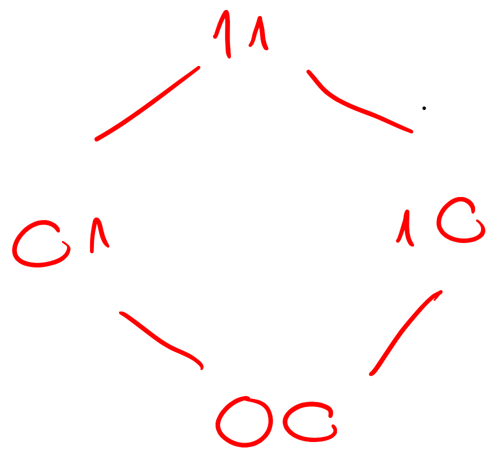

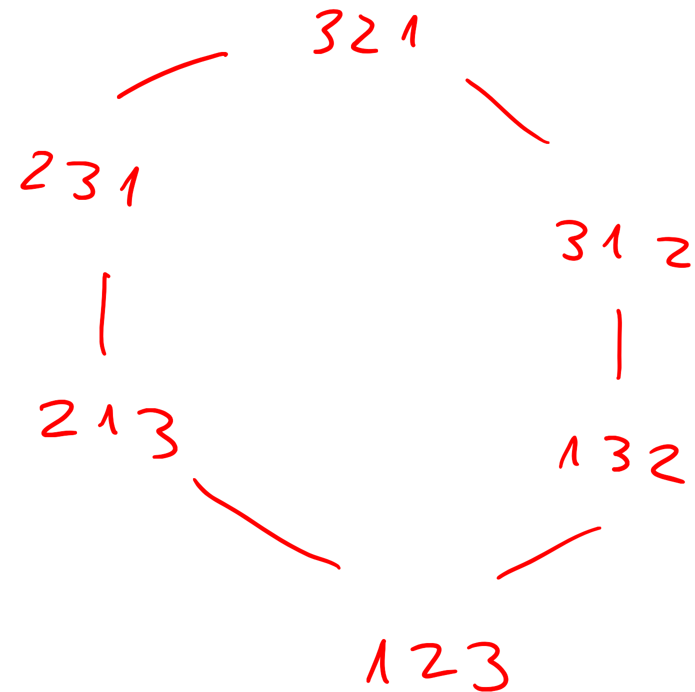

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Flip graph has Hamilton ~~cycle~~ path $\iff$ Existance of Gray code

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- **This talk:** the Greedy approach.

Greedy Gray codes

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- **Greedy approach:** Perform *locally optimal* move. Don't worry too much about the future.
- **Gray codes:** [Williams 13]
  - Rank the flips.
  - **Repeat:** Perform the highest ranked flip such that a new object is generated.

Examples

- n-bitstrings by bitflips.

0 0 0
0 0 1
0 1 1
0 1 0
1 1 0
1 1 1
1 0 1
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Examples

- n-bitstrings by bitflips.
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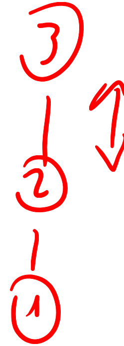

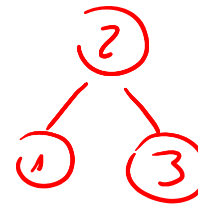

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- permutations of $[n]$ by transpositions.

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Not the traditional viewpoint.

A bit of discussion

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- Why do they work?
  - cf. the theory of matroids in optimization.

Zig-zag framework

Introduction

- Zig-zag framework **[Hartung, Hoang, Mütze, Williams 20]**

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- **Main idea:** Encode combinatorial objects as a set $F_n \subseteq S_n$ of permutations of length n.
- **Beyond permutations:** Just be inspired by them.

A simple algorithm

- **Start** with the identity object.

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- What is the difference with the generic greedy?
- **Reminder:** SJT

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1 2 4 3
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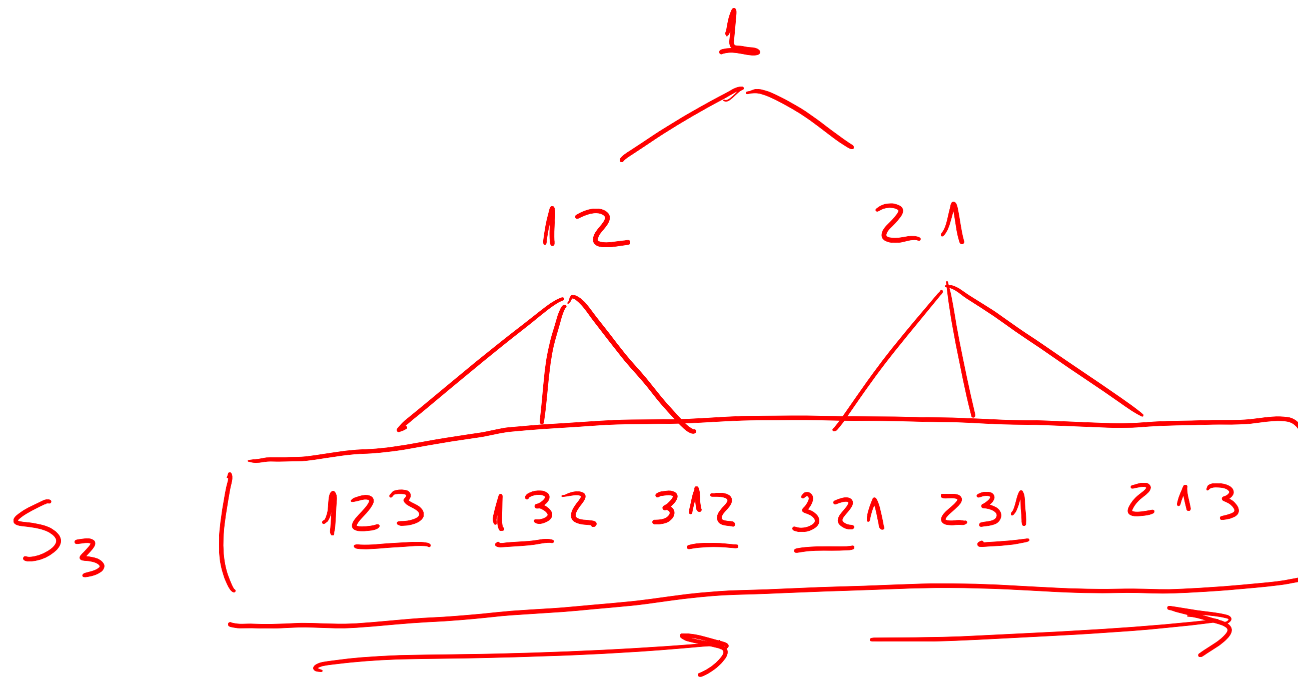
Tree of permutations for SJT

Idea: Build a recursive generation tree



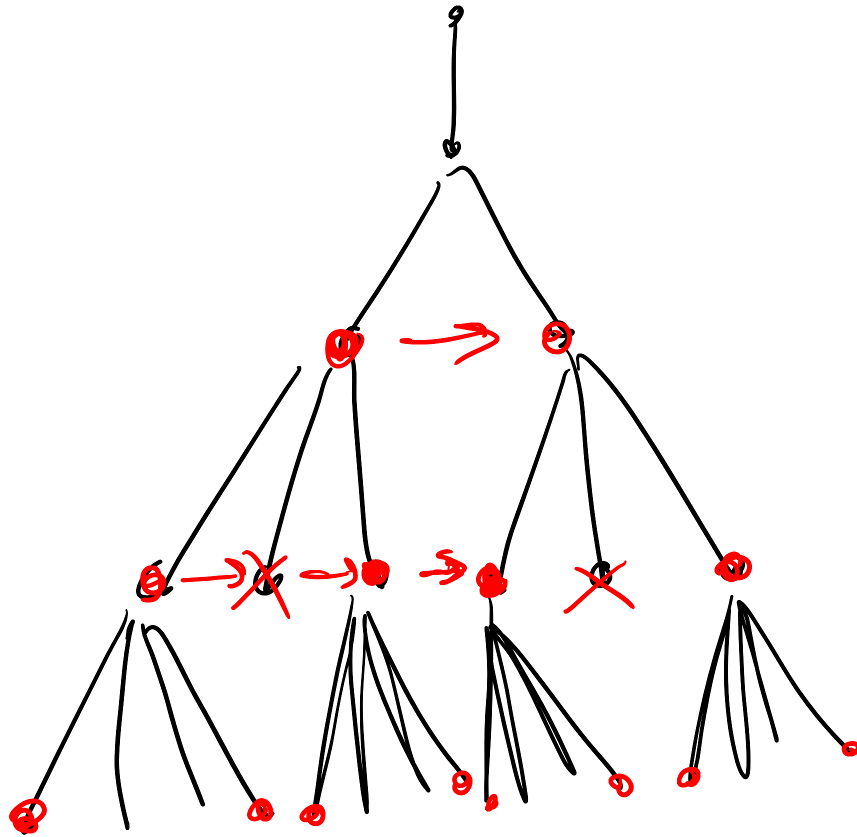
Tree of permutations for SJT

Key insight: The tree of permutations and the zig-zag algorithm produce the same listings.

Proof: Double induction

Depth + left to right.

Pruning



Zig-zag languages

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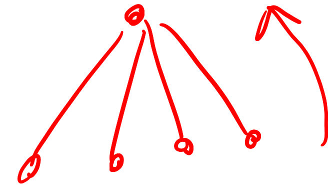

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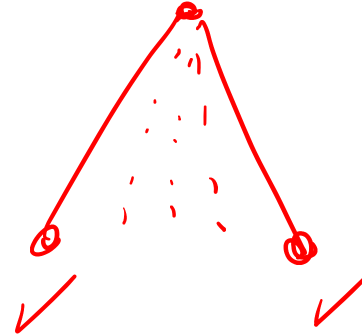

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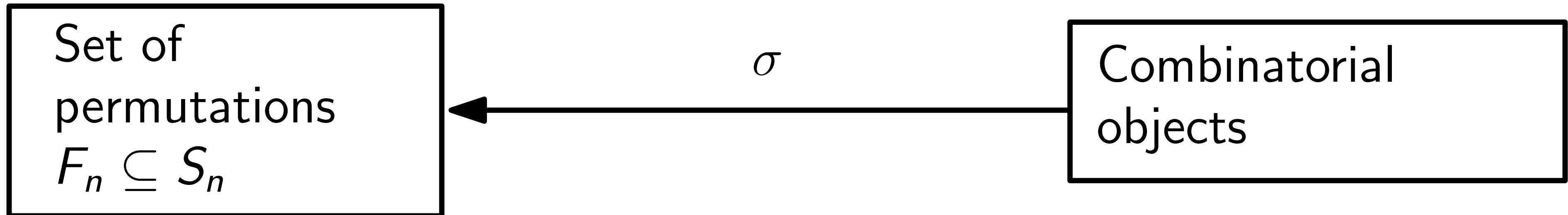
Very compatible with pattern-avoidance

General approach

Combinatorial objects

General approach



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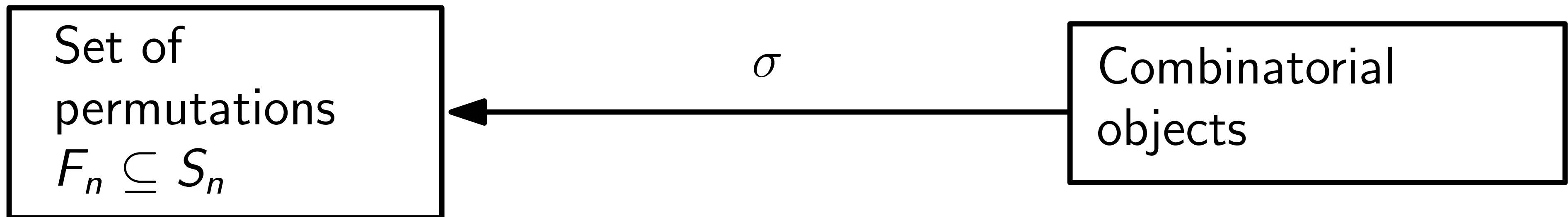
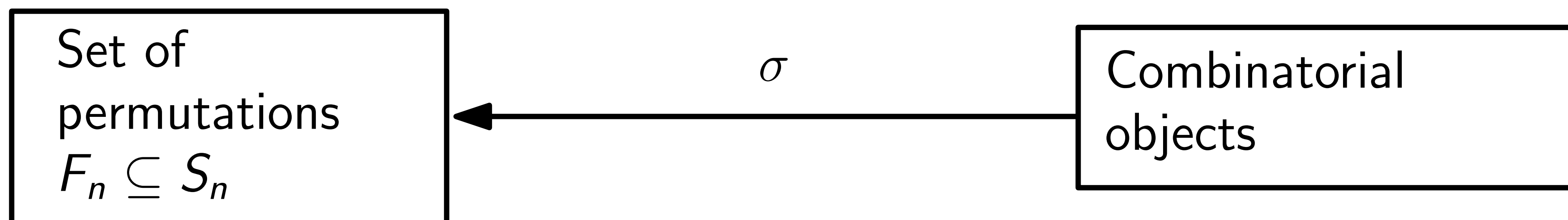


- run zig-zag algorithm

List $=$ **ZZ**(F_n) $\longrightarrow$ σ^{-1}(**List**)

General approach



- run zig-zag algorithm

$$\textbf{List} = \textbf{ZZ}(F_n) \longrightarrow \sigma^{-1}(\textbf{List})$$

- interpret zig-zag algorithm under the bijection

$$\textbf{ZZ} \longrightarrow \sigma^{-1}(\textbf{ZZ})$$

Towards efficiency: Memoryless

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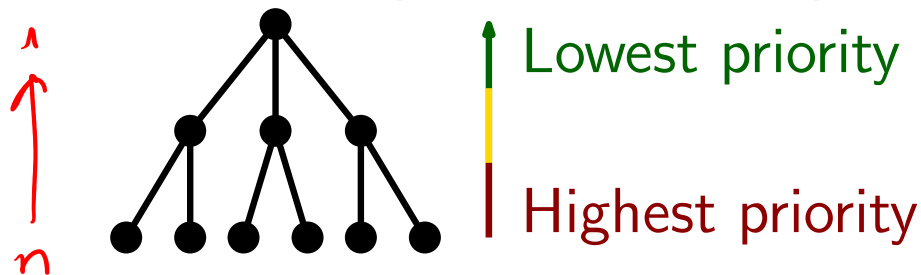

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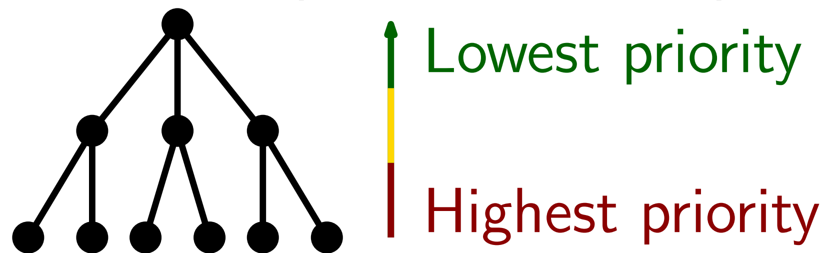
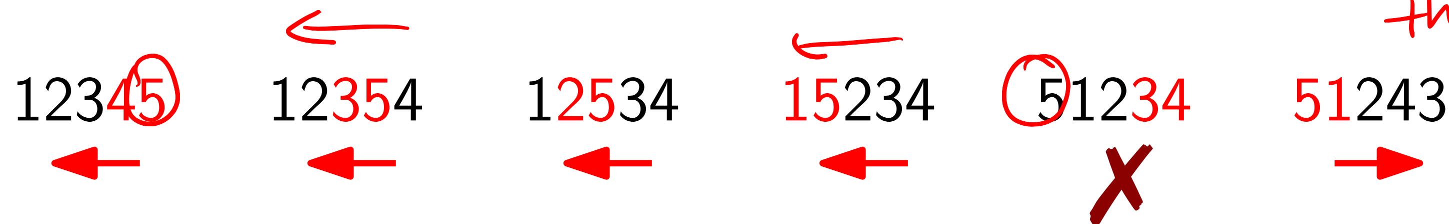


who is moving?

- Elements change direction when they cannot be moved.

where are they moving?



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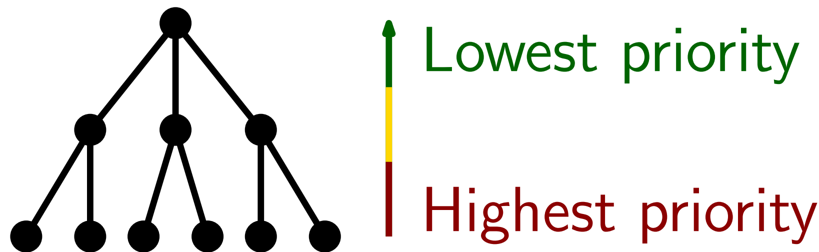
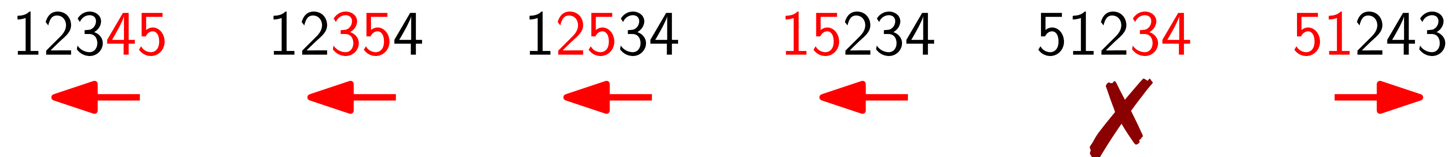


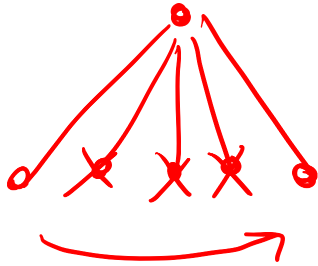
- Elements change direction when they cannot be moved.



- Memoryless version of the algorithm!

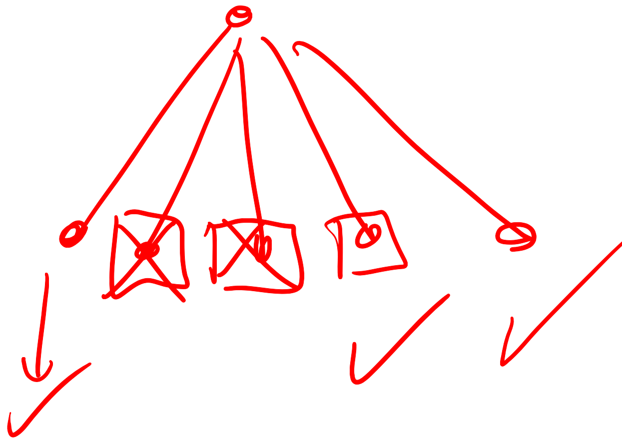
Towards efficiency: membership testing

- We only need to detect pruned branches.



Towards efficiency: membership testing

- We only need to detect pruned branches.
- *Difference* membership testing.



Without permutations: another view-point

- Try to *directly* construct a tree of the objects.

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- *Spiritual* zig-zag condition.
    - closed under deletion of the largest element.
    - closed under insertion at the beginning and at the end.

Examples: BRGC and Binary trees

BRGC:

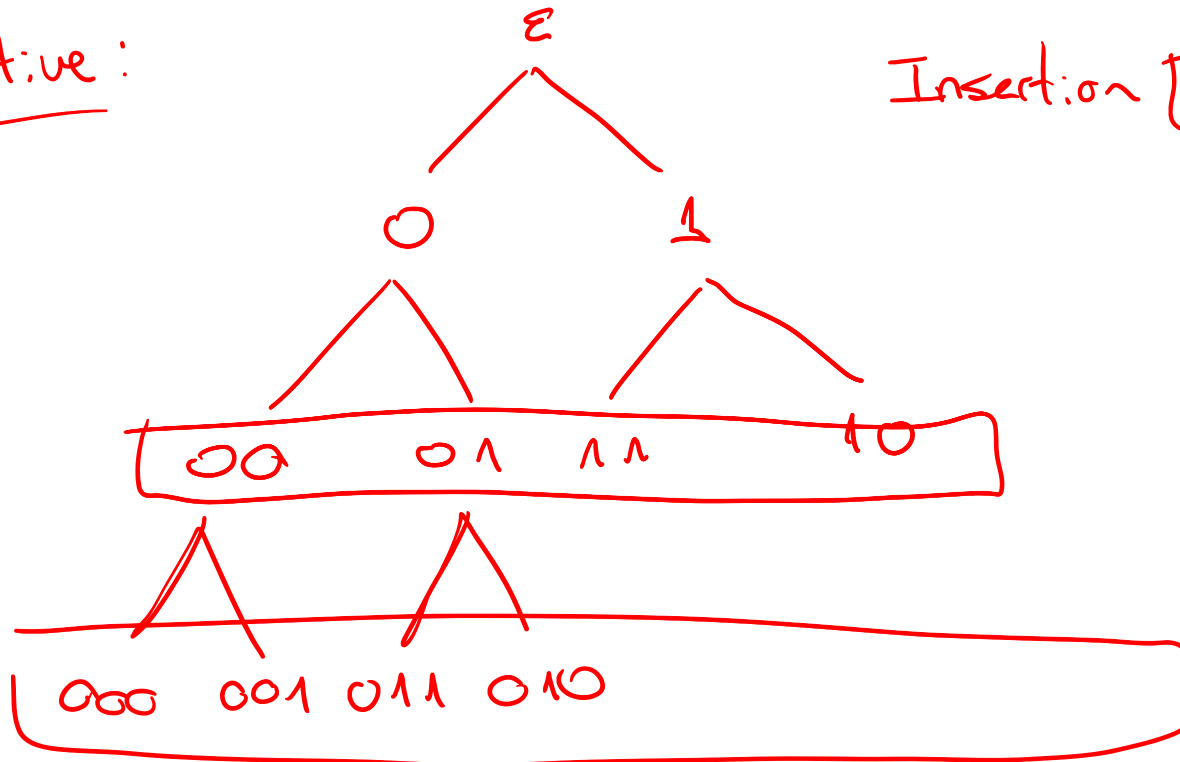
Main approach:

$\{0,1\}^n \hookrightarrow$ Peakless permutations

Deletion 0000 ✗

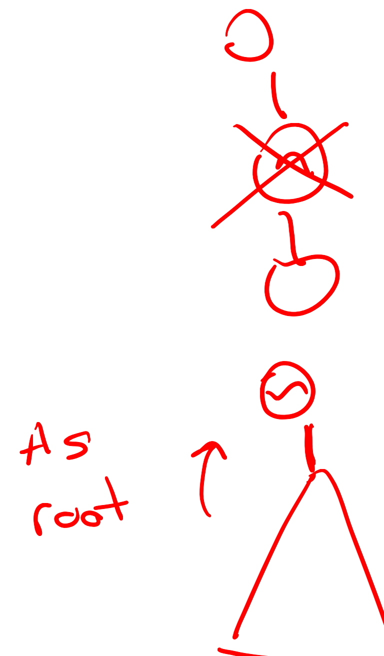
Insertion 0000 0 1

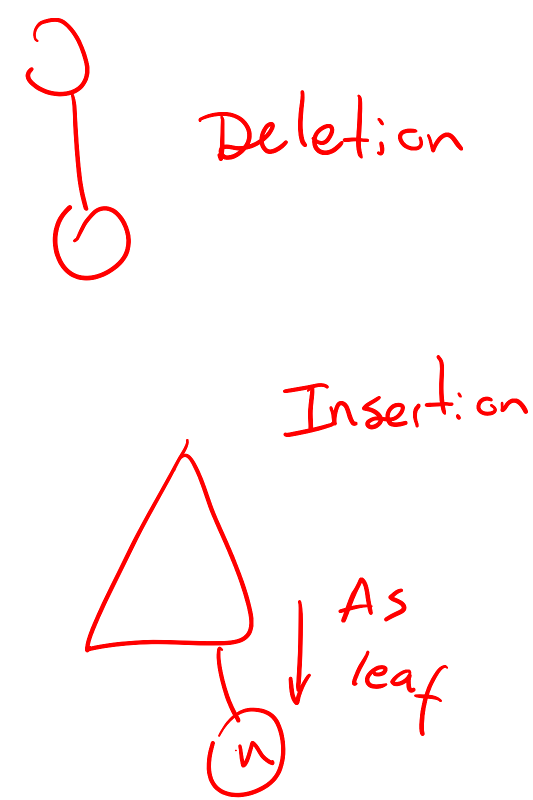
Alternative:



Examples: BRGC and Binary trees

Binary trees:





Non-examples (?)

non-necessarily adjacent.

- permutations of $[n]$ by transpositions.

Non-examples (?)

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- permutations of $[n]$ by prefix-reversal.

1 2 3 4 5

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Non-examples (?)

- permutations of $[n]$ by transpositions.
- permutations of $[n]$ by prefix-reversal.
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- *index-based* Gray codes.

Prefix-based Gray codes

???

A simple algorithm

- **Start** with any object.

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- What is the difference with the generic greedy? Zig-zag?

Examples

- permutations of $[n]$ by transpositions.
- permutations of $[n]$ by prefix-reversal.
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Examples

- permutations of $[n]$ by transpositions.
- permutations of $[n]$ by prefix-reversal.
- n pancakes with a burnt side by flips.
- n-bitstrings by prefix-complementation

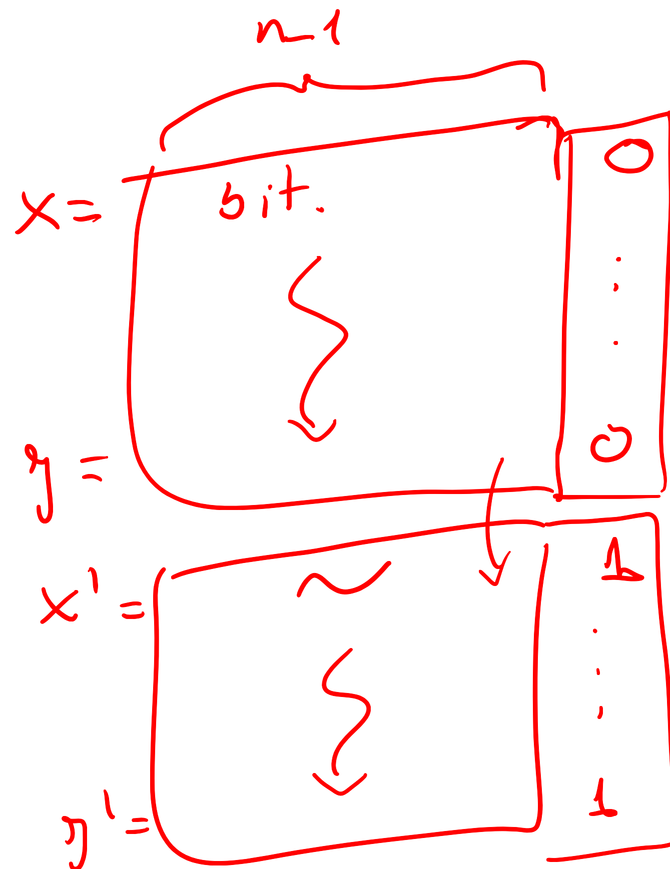
000
100
010
110
001

Proof: prefix-complementation.

By induction:

Hip: This alg. always works on n-bitstrings when starting w/ any bitstring.

induction gives everything on $n-1$-bitstrings

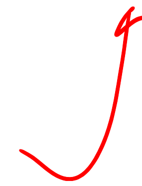


induction gives everything on $n-1$-bitstrings

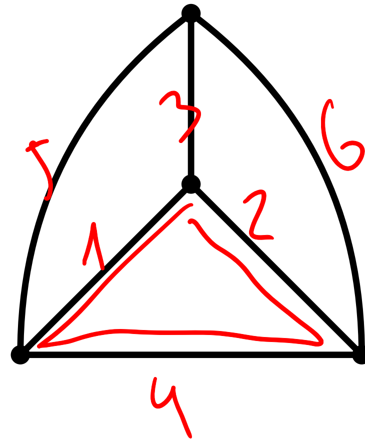
New: a greedy Gray code for spanning trees

- **Label** the edges in any order.
- **Start** with any spanning tree.
- **Repeat:** perform the flip $T \to T + i - j$ such that:
    - it generates a new object from the class, and
    - $\max\{i, j\}$ is maximized and break ties by minimizing $\min\{i, j\}$.

Doesn't really matter



Example: Spanning trees



1 2 3 $\longleftrightarrow$ 1 1 1 0 0 0

0 1 1 1 0 0

1 0 1 1 0 0

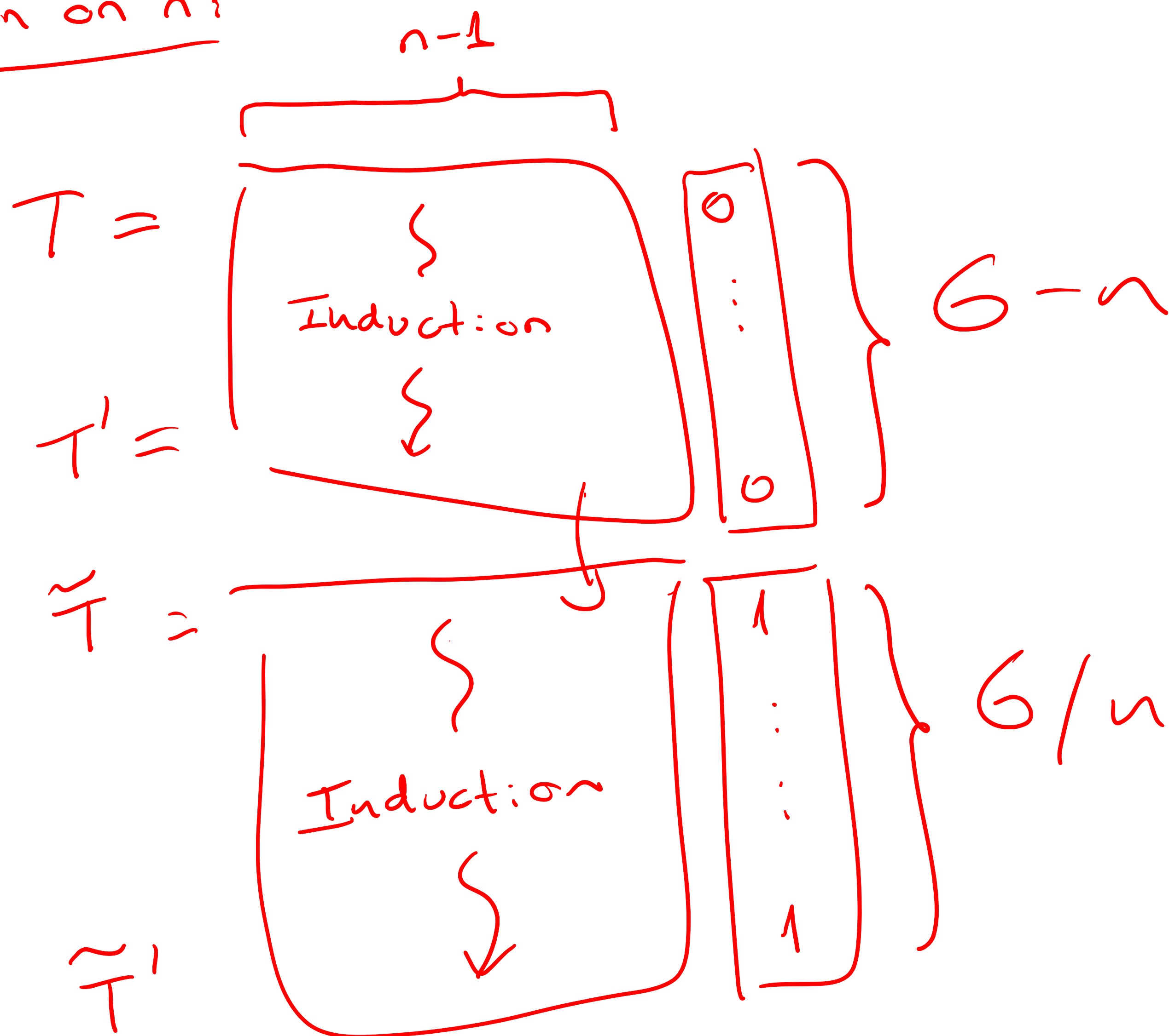
etc...

Perform shortest

Prefix edge-exchange

Proof: Spanning trees

Induction on n:



Some surprises/remarks about the spanning tree algorithm

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- Works for matroids!

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Final remarks

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- **Link (?):** inversion table.

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- New matroid result.

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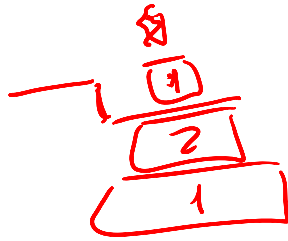

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- Better understanding of prefix-based greedy Gray codes.

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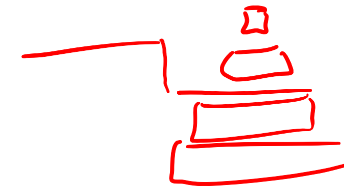

- **Concrete:** $\mathcal{O}(1)$ delay for spanning trees.
  - **Known:** Amortized $\mathcal{O}(1)$ delay.

Thanks!