Computing the Maximum Function with ReLU Neural Networks

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Joint WIP with
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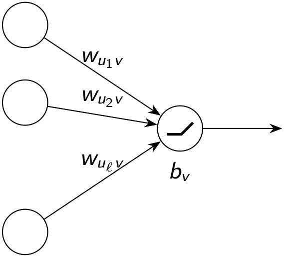
IOL & COGA Research Seminar
November 5, 2020

A Single (Hidden) ReLU Neuron

Outputs $o(u_i)$ of previous neurons



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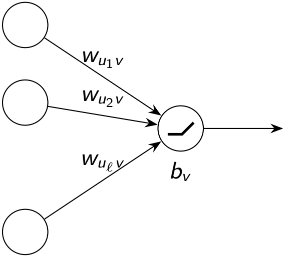

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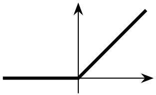
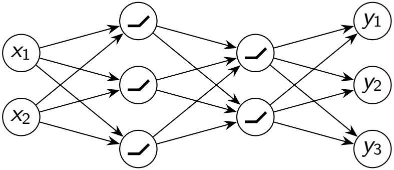
Rectified linear unit (ReLU): $\mathrm{relu}(x) = \max\{0, x\}$



ReLU Feedforward Neural Networks

- Acyclic (layered) digraph of ReLU neurons



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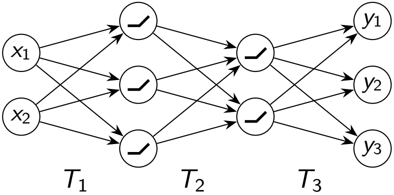

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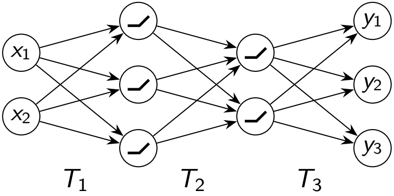

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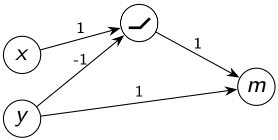
- Example: depth 3 (2 hidden layers), width 3.

Example: Computing the Maximum of Two Numbers

$$\max\{x, y\} = \max\{x - y, 0\} + y$$



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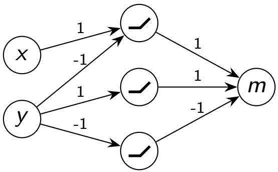

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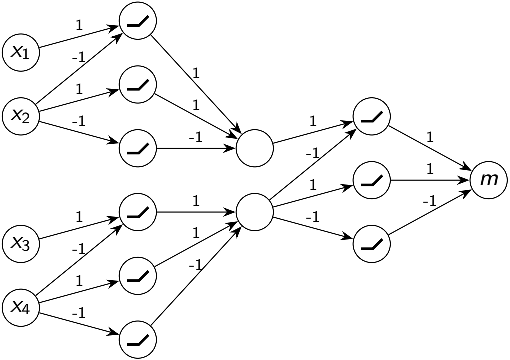

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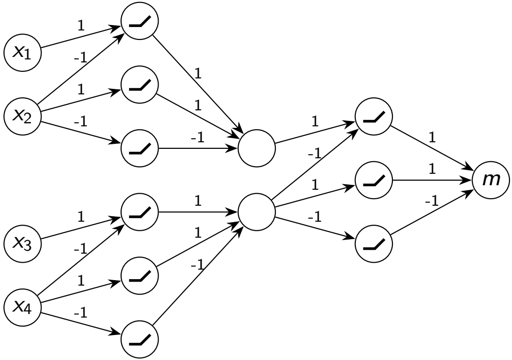

- Inductively: Maximum of n numbers with depth $\lceil \log_2(n) \rceil + 1$.

Expressivity of ReLU neural networks

Theorem (Wang, Sun (2005))

For any continuous and piecewise linear (CPWL) function $f : \mathbb{R}^n \to \mathbb{R}$, there are $\alpha_i \in \mathbb{R}$, $a_{ij} \in R^n$, and $b_{ij} \in \mathbb{R}$ such that

$$f(x) = \sum_i \alpha_i \max\{a_{ij}^T x + b_{ij} \mid j = 1, \dots, n+1\}.$$

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In this case, depth $\lceil \log_2(n+1) \rceil + 1$ suffices.

$\Rightarrow$ Everything depends on the maximum function!

Is logarithmic depth best possible?

- Known: $\max\{0, x_1, x_2\}$ cannot be computed with 2 layers.

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  Can $\max\{0, x_1, x_2, x_3, x_4\}$ be computed with 3 layers?
- No function known that provably needs more than 3 layers.

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(for notational purposes: $x_0 := 0$.)

Strategy:

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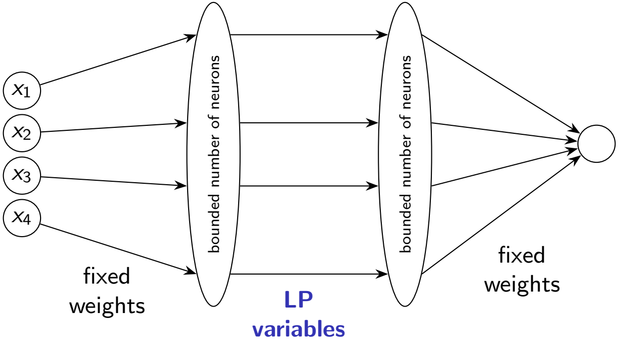
Construct a linear program (LP) that is feasible if and only if such a neural network exists.

Rough Idea



x_1
x_2
x_3
x_4
bounded number of neurons
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fixed weights
LP variables
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$\Rightarrow$ From now on, only consider NNs without biases.

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The output of each neuron can only have breakpoints when the relative ordering of the five numbers 0, x_1, ..., x_4 changes.

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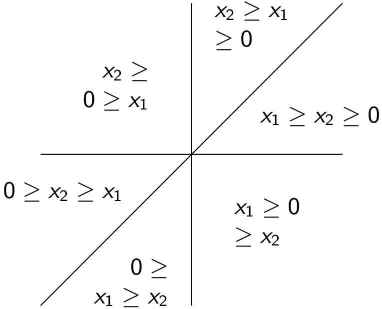

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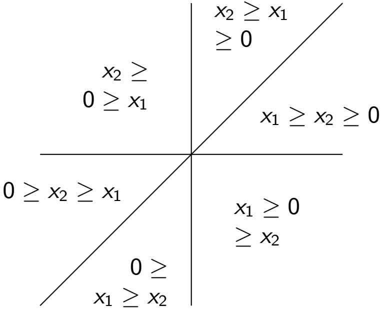
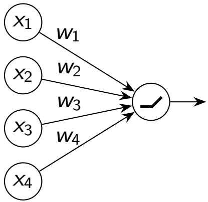


The $\binom{5}{2} = 10$ hyperplanes $x_i = x_j$, $0 \leq i < j \leq 4$, subdivide $\mathbb{R}^4$ into $5! = 120$ cells in each of which the output of each neuron is affine.

Neurons in the First Hidden Layer



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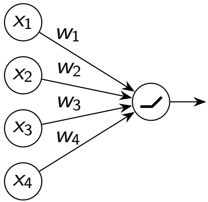

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- Must be one of the 10 specified hyperplanes.

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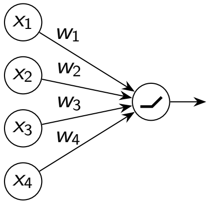

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Claim: The output of a neuron in the first hidden layer is a linear combination of the following 14 functions:

$$\begin{array}{cccc}
\max\{x_1, 0\}, & \max\{x_2, 0\}, & \max\{x_3, 0\}, & \max\{x_4, 0\}, \\
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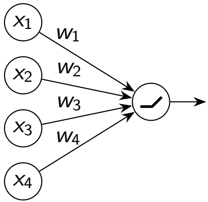

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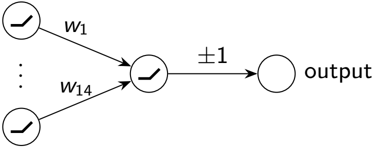
$\Rightarrow$ **14 neurons suffice!**

Neurons in the Second Hidden Layer

Observe: WLOG output weights ± 1.



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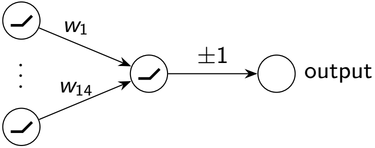

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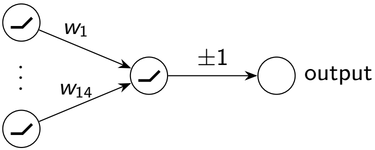

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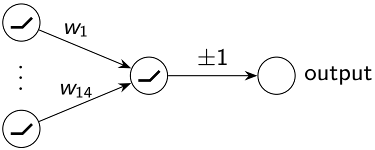

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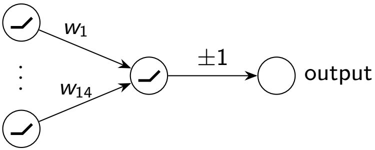

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$$\Rightarrow \mathbf{2^{121}} \textbf{ neurons suffice!}$$

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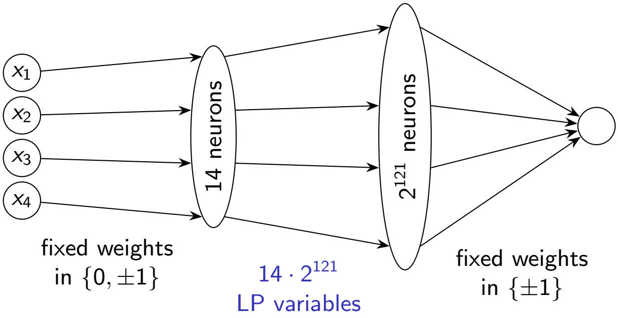
$\rightsquigarrow$ For this talk suppose $2^{121} \approx 200\,000$.

Variables of the Linear Program



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Two types of constraints:

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$\Rightarrow$ Have these constraints:

1. $4 \cdot 120$ equality constraints.
2. $4 \cdot 120 \cdot 2^{121}$ inequality constraints.

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... for less than a minute ...

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... and outputs ...

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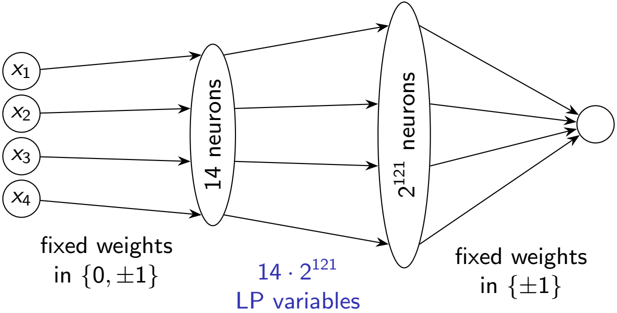
Infeasible!

Three Obvious Next Steps ...

- Proving the assumption.
- Finding a "real" proof.
- Generalize to more layers.

Thank you!



Questions? Ideas?