Robust Optimization and Learning

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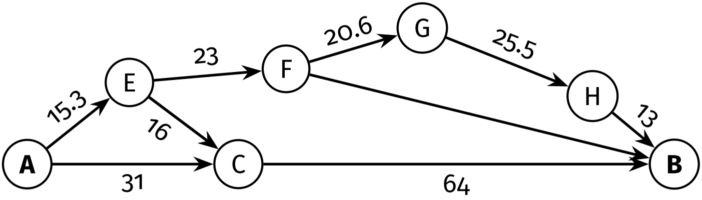
joint work with Kevin-Martin Aigner, Kristin Braun, Frauke Liers and Sebastian Pokutta

What is the problem?



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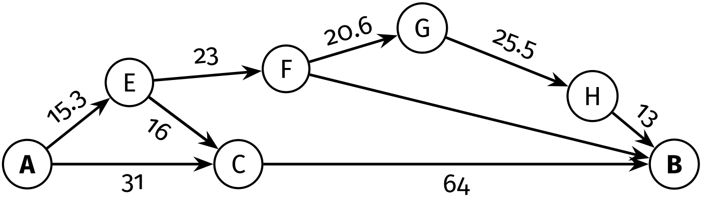

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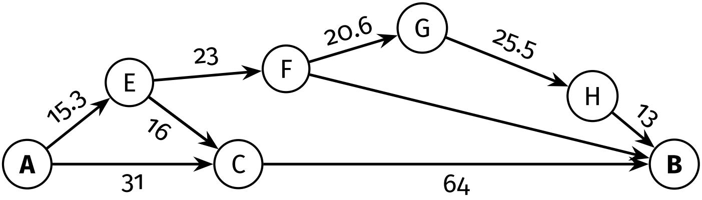

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$$\min_{\mathbf{y}} \ \mathbb{E}_{\boldsymbol{\xi} \in \mathcal{U}}[\mathbf{d}(\boldsymbol{\xi})^\top \mathbf{y}]$$

$$\text{s.t. } \mathbf{y} \in Y$$

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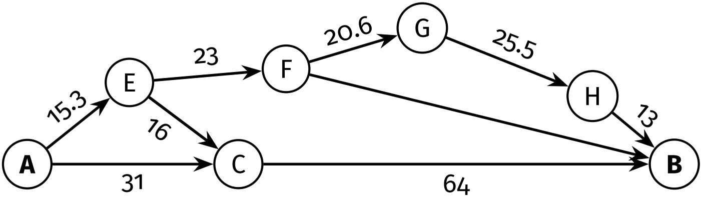

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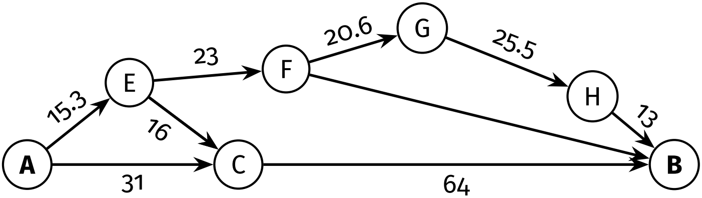

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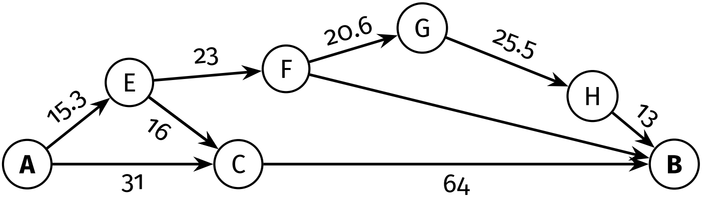

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Distributionally Robust Optimization

$$\min_{x \in \mathcal{X}} \max_{p \in \mathcal{P}} \mathbb{E}_{s \sim p}[f(x, s)]$$

- $\mathcal{P}$ referred to as the ambiguity set
- Multiple ways to define the set ($\mathcal{M}$ bet set of probability measures)
    - Moment Constraints $\mathcal{P} = \{p \in \mathcal{M} \mid |\mathbb{E}_p[s] = \mu, \mathbb{E}_p[s^2] = \sigma\}$
    - Distance Constraints $\mathcal{P} = \{p \in \mathcal{M} \mid |d_w(p, p_0) \leq \epsilon\}$

Model

Assumptions

- Finite scenario set
- Limited knowledge about scenario probabilites
- Over time the parameters l^t and u^t which are bounds on the probability estimate change.

$$\min_{x \in \mathcal{X}} \max_{p} \sum_{s \in \mathcal{S}} f(x, s) p_s$$

$$\sum_{s \in \mathcal{S}} p_s = 1$$

$$l^t \leq p \leq u^t$$

$$p \geq 0.$$

Probability Bounds

Confidence Intervals: Multinomial Distribution

- χ^2-estimator:

$$[l_i^t, u_i^t] = \left[\frac{n_i - c}{t}, \frac{n_i + c}{t} + \frac{2\gamma}{t}\right]$$

- Maximum Likelihood estimator:

$$[l_i^t, u_i^t] = \left[\frac{2n_i + \phi - \sqrt{\delta_i}}{2(\phi + t)}, \frac{2n_i + \phi + \sqrt{\delta_i}}{2(\phi + t)}\right]$$

where $\phi = f\left(1 - \frac{\alpha}{k}\right), \quad \delta_i = \phi^2 + (4n_i\phi)\left(1 - \frac{n_i}{t}\right)$

Algorithm

Possible Methods

- Direct Solution
    - Use duality theory to reformulate the inner maximization problem
- Online Learning
    - Alternate between the inner maximization and outer minimization problems

Algorithm: Direct Solution

Dual Reformulation

$$\begin{aligned}\min_{x,z,\alpha,\beta}\ & \tau \\ & z - (l^t)^T\alpha + (u^t)^T\beta \leq \tau \\ & f(x,s) - z + \alpha_s - \beta_s \leq 0, \forall s \in \mathcal{S} \\ & \alpha, \beta \geq 0 \\ & x \in \mathcal{X}.\end{aligned}$$

Algorithm: Online Learning

Player x

- Input: p_t
- solve $\min_{x \in X} \mathbb{E}[f(x,s)]$ with solution x^t
- Output: x^t

Player p

- Input p_t, x_t
- Make a projected mirror descent step as per the following
- $p_{t+1} = \arg\min_{p \in \mathcal{P}^t} \langle -\eta^t \nabla_p \mathbb{E}_{s \sim p^*}[f(x^t, s)], p \rangle + V_{p^t}(p)$ with $V_{p^t}(p) = \sum_{s \in \mathcal{S}} p_s \log \left(\frac{p_s}{p_s^t} \right)$

Theoretical Results

Assumptions

- Convexity
- Each uncertainty set $\mathcal{P}_t \subseteq \mathcal{P}_{t-1}$

Theorem

$$\max_{p \in \mathcal{P}_T} \mathbb{E}_{s \sim p}[f(\bar{x}, s)] - \min_{x \in \mathcal{X}} \frac{1}{T} \sum_{t=1}^{T} \max_{p_t \in \mathcal{P}_t} \mathbb{E}_{s \sim p_t}[f(x, s)] \leq \frac{R_p(T) + R_x(T)}{T}$$

Proof

Steps

- Prove regret bounds for each optimization step

$$\sum_{t=1}^{T} \mathbb{E}_{s\sim p_t}[f(x_t, s)] - \min_{x\in\mathcal{X}} \sum_{t=1}^{T} \mathbb{E}_{s\sim p_t}[f(x, s)] \leq R_x(T)$$

$$\max_{p\in\mathcal{P}_T} \sum_{t=1}^{T} \mathbb{E}_{s\sim p}[f(x_t, s)] - \sum_{t=1}^{T} \mathbb{E}_{s\sim p_t}[f(x_t, s)] \leq R_p(T)$$

- Combine the above equations

Numerical Experiments

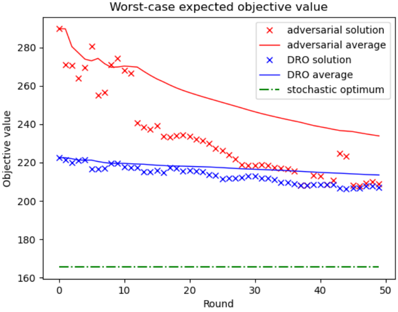
Experiments

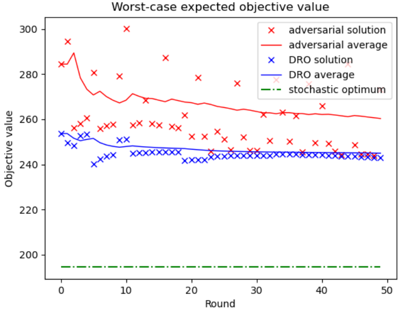
- Quadratic progams
- Network Flow problems

Setup

- Python 3.8 and Gurobi 9.1
- At each time we report
  - Optimal objective for reformulation
  - Worst case for the current solution of the online problem

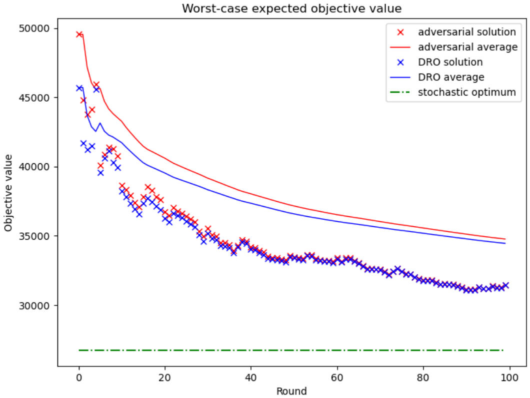
Numerical Experiments: Quadratic progams





- Convex Quadratic Program (QPLIB 3871)
- Avg Time for RO problem 28.41s, 46.95s
- Avg Time for Adversarial problem 12.12s, 15.89s

Numerical Experiments: Network Flow problems

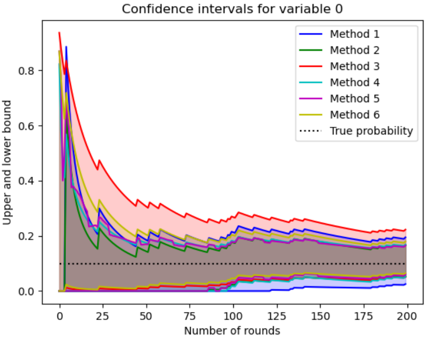


- Multi Commodity Network Flow problem
- Avg Time for RO problem 0.54s
- Avg Time for Adversarial problem 0.14s

Conclusion

- The online learning method is faster however it takes many rounds to achieve comparable solution quality
- The numerical results verify the theoretical counterparts indicating decreasing regret.
- The assumptions for the theoretical results are very strict

Confidence Interval Methods



- Gold (1963)
- Goodman (1965)
- Quesenberry (1964)
- Fitzpatrick (1987)
- Sison-Glaz (1995)
- Python-Implementation Quesenberry (1964)

Sublinear Regret